Fluctuations and light scattering in a liquid crystal polymer fiber

R. A. Sones and R. G. Petschek
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106-7079
(Received 17 January 1995; revised manuscript received 26 October 1995)

The spatial correlation function is calculated for thermal fluctuations of the liquid crystal director field in the core of an optical fiber composed of a main chain liquid crystal polymer in the smectic-C phase. Due to steric constraints on the liquid crystal polymer, the smectic layers orient perpendicular to the fiber axis, the liquid crystal director lies flat against the core-cladding interface, and a line disclination appears along the fiber axis. The analysis is based on a Landau–de Gennes expression for free energy, which includes ordinary smectic-C elastic terms as well as terms added to model the disclination. The correlation function is used to relate the scattering distribution and attenuation of guided light to the elastic constants of the liquid crystal polymer.

PACS number(s): 61.30.Cz, 42.81.Dp, 42.81.Wg, 61.30.Jf

I. INTRODUCTION

The elastic constants and dynamics of liquid crystal polymers are of fundamental interest. One standard probe of these properties, light scattering, is best suited for well-aligned samples, and such samples are difficult to obtain with liquid crystal polymers. Fortunately, a main chain polymer material can be well aligned by drawing (stretching) it into a fiber [1], but, unfortunately, deforming the material this way complicates the interpretation of light scattering measurements. In this paper we discuss a common fiber geometry, namely, that of an optical fiber—with a main chain smectic-C (denoted Sm-C) liquid crystal polymer (LCP) core surrounded by an isotropic cladding—and show how the LCP elastic constants are related to the scattering distribution from light guided by the fiber.

Optical fibers of this sort may be strongly attenuating—since light scatters from fluctuations in the dielectric tensor caused by thermal fluctuations in the liquid crystal director field—and this could potentially prevent guided-light scattering measurements from being practical [2–6]. Hence, we also estimate the attenuation of such fibers. Since there has been recent interest in the construction of optical fibers with liquid crystal cores [2,7], our calculations may soon have experimental relevance.

The fiber geometry is illustrated in Fig. 1. We assume the LCP mesogens lie flat against the core-cladding interface, with the smectic layers oriented perpendicular to the fiber axis and a line disclination along the fiber axis. (The motivation for these assumptions is given at the beginning of the next section.) Our goal is to calculate the spatial correlation function for thermal fluctuations of the liquid crystal director field, and from this estimate the scattering distribution and attenuation of the fiber.

The paper is organized as follows. In Sec. II we write a Landau–de Gennes free energy of the fiber core, minimize to find the equilibrium configuration, and calculate the normal modes of thermal fluctuations about equilibrium. In Sec. III we use the normal mode expansion to calculate the spatial correlation function. In Sec. IV we use

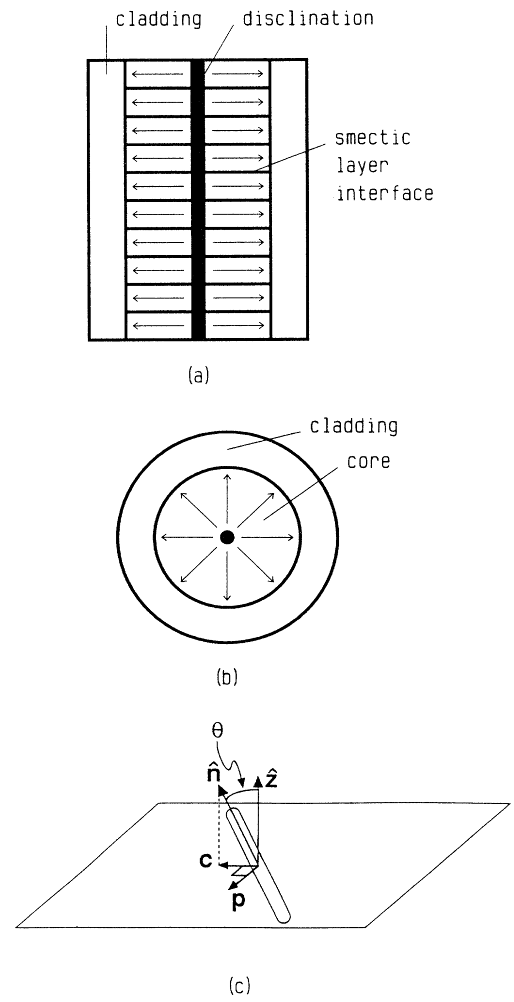

FIG. 1. (a) Fiber cross section containing the fiber axis. The disclination runs along the fiber axis, perpendicular to the smectic layers. Arrows depict the **p** field. (b) Fiber cross section perpendicular to the fiber axis. Arrows depict the **p** field. (c) Relationship between the layer normal $\hat{\mathbf{z}}$, director $\hat{\mathbf{n}}$, **p**, and **c**.

the correlation function and classical scattering theory to calculate the scattering distribution and attenuation of the fiber. In Sec. V we discuss our results.

II. FREE ENERGY

Consider an optical fiber with a core of radius R and length $L \to \infty$, and let the axis of the fiber coincide with the z axis. Adopt standard cylindrical coordinates (ρ, ϕ, z). The core is composed of main chain LCP in the Sm-C phase, and we assume the smectic layers are ideal (i.e., rigid and uniformly spaced) and oriented perpendicular to $\hat{\mathbf{z}}$ (see Fig. 1). Let $\hat{\mathbf{n}}$ be the liquid crystal director, which tilts slightly away from $\hat{\mathbf{z}}$, and let $\mathbf{c}$ be the component of $\hat{\mathbf{n}}$ parallel to the layers. Define the vector $\mathbf{p} = \hat{\mathbf{z}} \times \mathbf{c}$, which is parallel to the layers and perpendicular to both $\hat{\mathbf{n}}$ and $\mathbf{c}$. The vector field $\mathbf{p}$ (which is everywhere in the xy plane) will serve as the Sm-C order parameter.

The motivation for adopting this geometry [i.e., with $\mathbf{p}(\mathbf{r})$ radial and a 2π disclination [8–11] along the fiber axis] is as follows. We assume the core and cladding materials have been chosen so that the surface tension at the core-cladding interface is minimized when $\mathbf{p}$ is normal to the interface. This is known to be possible for low-molecular-weight Sm-C liquid crystals. Since the energy cost of the disclination grows slowly (logarithmically) with the radius of the fiber core while the core-cladding interface energy grows linearly, it is expected that, at least for large radii (as observed experimentally in a physically similar situation [12]), the interface energy will dictate the orientation of $\mathbf{p}$ at the fiber surface.

It has been shown that, due to steric constraints [13], surfaces containing the director of a main chain nematic liquid crystal are likely to have significantly lower free surface tension than other surfaces. Similar behavior is expected for main chain Sm-C liquid crystals, so that *at least* one direction perpendicular to the order parameter $\mathbf{p}$ (such as $\hat{\mathbf{n}}$ or $\mathbf{c}$) is expected to lie in a low-energy surface. It is conceivable that *both* directions perpendicular to $\mathbf{p}$ should be expected to lie in a low-energy surface, which would induce $\mathbf{p}$ to be perpendicular to the surface. These considerations are consistent with our assumption that $\mathbf{p}$ is normal to the core-cladding interface, and suggest that the choice of the core and cladding materials may not be critical.

Write the free energy of the fiber core as [14,15]

$$F = \int d\mathbf{r} \left\{ \frac{K_1}{2} (\hat{\mathbf{z}} \cdot \nabla \times \mathbf{p})^2 + \frac{K_2}{2} \left(\frac{\partial \mathbf{p}}{\partial z} \right)^2 + \frac{K_3}{2} (\nabla \cdot \mathbf{p})^2 - Ap^2 + Bp^4 \right\}, \tag{1}$$

where the Ks are elastic constants with units of force (we neglect the K_4 term, which is typically small), A and B are elastic constants associated with the Sm-C tilt angle, and $\mathbf{p}(\mathbf{r})$ is allowed to vary in magnitude as well as orientation. The stability of the Sm-C structure imposes the constraints $K_1, K_2, K_3 > 0$, and, in a Sm-C LCP, one expects $K_2 \sim K_3$ and the ratio K_1/K_2 to be of the order of the degree of polymerization [16–20]. Here, to simplify the analysis, we let $K_2 = K_3 = K$. In the neighborhood of the disclination $p \to 0$, otherwise the gradient terms diverge. Far from the disclination the gradient terms vanish and

$$p \to \sqrt{\frac{A}{2B}} = \sin\theta_0 ,$$

where θ_0 is the value of the Sm-C tilt angle in bulk material (typically zero to 30°). Let

$$\mathbf{p}(\mathbf{r}) = p_0(\rho)\, \hat{\boldsymbol{\rho}} + \mathbf{p}_1(\mathbf{r}) , \tag{2}$$

where the first term is the equilibrium (zero temperature) solution and the second term represents fluctuations from equilibrium. Grouping terms in the free energy that are of zeroth, first, and second order in p_1 give

$$F_0 = \int d\mathbf{r} \left\{ \frac{K}{2} \left(\frac{p_0}{\rho} + p_0' \right)^2 - Ap_0^2 + Bp_0^4 \right\}, \tag{3}$$

$$F_1 = \int d\mathbf{r} \left\{ K \left(\frac{p_0}{\rho} + p_0' \right) \nabla \cdot \mathbf{p}_1 - (2Ap_0 - 4Bp_0^3)\, \hat{\boldsymbol{\rho}} \cdot \mathbf{p}_1 \right\}, \tag{4}$$

$$F_2 = \int d\mathbf{r} \left\{ \frac{K_1}{2} (\hat{\mathbf{z}} \cdot \nabla \times \mathbf{p}_1)^2 + \frac{K}{2} \left(\frac{\partial \mathbf{p}_1}{\partial z} \right)^2 + \frac{K}{2} (\nabla \cdot \mathbf{p}_1)^2 - Ap_1^2 + 4Bp_0^2 \left(\frac{p_1^2}{2} + (\hat{\boldsymbol{\rho}} \cdot \mathbf{p}_1)^2 \right) \right\}, \tag{5}$$

where $F = F_0 + F_1 + F_2 + O(p_1^3)$, the region of integration is the fiber core, and the primes denote differentiation.

In equilibrium $F_1 = 0$, and this will determine the form of the equilibrium solution $p_0(\rho)$. Integrating F_1 by parts gives

$$F_1 = \int d\mathbf{r} \left\{ -K \left(p_0'' + \frac{p_0'}{\rho} - \frac{p_0}{\rho^2} \right) - 2Ap_0 + 4Bp_0^3 \right\} \hat{\boldsymbol{\rho}} \cdot \mathbf{p}_1 , \tag{6}$$

where we neglect surface terms. To satisfy $F_1 = 0$ the factor in braces in the integrand must be zero. With the change of variables $u = \rho/R_d$, $v(u) = p_0/p_\infty$ we find

$$u^2 v'' + uv' + [2\left(1 - v^2\right) u^2 - 1]v = 0 , \tag{7}$$

where $R_d = K^{1/2} A^{-1/2}$ is a length that characterizes the radius of the disclination, and

$$p_\infty = \sqrt{\frac{A}{2B}} \tag{8}$$

is the asymptotic (bulk) value of p. As $u \to 0$ the solution that remains finite is $v(u) \sim u$, and as $u \to \infty$ we require $v(u) \to 1$. The solution for these boundary conditions was calculated numerically (see Fig. 2). The approximation

$$v(u) = (1 - 0.175\,\mathrm{sech}\,u)\tanh u \tag{9}$$

satisfies the boundary conditions and reproduces the numerical solution to within 2% for all u. Typically $R \sim 10^{-6}$ m and $R_d \sim 10^{-8}$ m, so the volume of the disclination is minute compared to the volume of the fiber core. Outside the disclination v rapidly approaches unity.

Now that we have determined the equilibrium configuration, we turn our attention to thermal fluctuations about equilibrium. To lowest nonvanishing order, the change in the free energy due to the fluctuation $\mathbf{p}_1(\mathbf{r})$ is F_2. We will transform the expression for F_2 into a summation over normal modes, and expand $\mathbf{p}_1(\mathbf{r})$ in terms of these normal modes—this will simplify the correlation function calculation in Sec. III. Since F_2 is second order in $\mathbf{p}_1$ we can, in principle, write it in the form

$$F_2 = KR_d \int \frac{d\mathbf{r}}{R_d^3}\,\mathbf{p}_1^\dagger M \mathbf{p}_1\,, \tag{10}$$

where M is a 2×2 matrix differential operator (with dimensionless elements M_{xx}, M_{xy}, M_{yx}, and M_{yy}) and $\dagger$ denotes the Hermitian conjugate. The eigenfunctions of M are the normal modes we seek. Since the fiber is cylindrically symmetric, M commutes with the infinitesimal translation and rotation generators

$$-i\frac{\partial}{\partial z},\quad -i\frac{\partial}{\partial \phi} + i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenfunctions of the translation generator are $e^{ik_\ell z}$ (times an arbitrary function of ρ and ϕ), where $k_\ell = 2\pi\ell/L$, ℓ is an integer (the axial quantum number), L is the length of the fiber, and we have adopted periodic boundary conditions in the z direction. The eigenfunctions of the rotation generator are $e^{im\phi}\hat{\boldsymbol{\rho}}$ (times an arbitrary function of ρ and z) and $e^{im\phi}\hat{\boldsymbol{\phi}}$ (times an arbitrary function of ρ and z), where m is an integer (the azimuthal quantum number). So we may write the eigenfunctions of M in the form

$$\Lambda = \left[f(u)\,\hat{\boldsymbol{\rho}} + ig(u)\,\hat{\boldsymbol{\phi}}\right]\, e^{im\phi}\, e^{ik_\ell z}\,, \tag{11}$$

where f and g give the radial dependence of the radial and azimuthal components of a mode. The eigenvalue equation $M\Lambda = \lambda\Lambda$ leads to the coupled ordinary differential equations [obtained by varying $\int d\mathbf{r}\Lambda^\dagger(M-\lambda)\Lambda$ with respect to f and g]

$$u^2 f'' + uf' + \left[u^2(\mathcal{E} + 2 - 6v^2) - (\gamma m^2 + 1)\right] f + (\gamma - 1)mug' + (\gamma+1)mg = 0\,, \tag{12}$$

$$u^2 g'' + ug' + \left[\frac{u^2}{\gamma}(\mathcal{E} + 2 - 2v^2) - \left(\frac{m^2}{\gamma} + 1\right)\right] g + \left(\frac{1}{\gamma} - 1\right) muf' + \left(\frac{1}{\gamma} + 1\right) mf = 0\,, \tag{13}$$

where

$$\lambda = \tfrac{1}{2}[\mathcal{E} + (k_\ell R_d)^2]\,, \tag{14}$$

and $\gamma = K_1/K$.

Both λ and $\mathcal{E}$ must be non-negative, since they are associated with fluctuations away from equilibrium. The dominant fluctuation modes are those which are most easily excited, i.e., with $0 \le \mathcal{E} \ll 1$. When $u \gg 1$ the differential equations decouple, and the dominant modes have $f \sim \exp(-2u)$ (the boundary condition $\mathbf{p}_1 \to 0$ as $u \to R/R_d$ demands the decaying exponential). Therefore, f is negligible except in the vicinity of the disclination, and it is reasonable to consider an approximation where we neglect f entirely. Then the coupled differential equations (12) and (13) simplify to

$$f = 0\,, \tag{15}$$

$$u^2 g'' + ug' + \left[\frac{u^2}{\gamma}\mathcal{E} - \left(\frac{m^2}{\gamma} + 1\right)\right] g = 0\,, \tag{16}$$

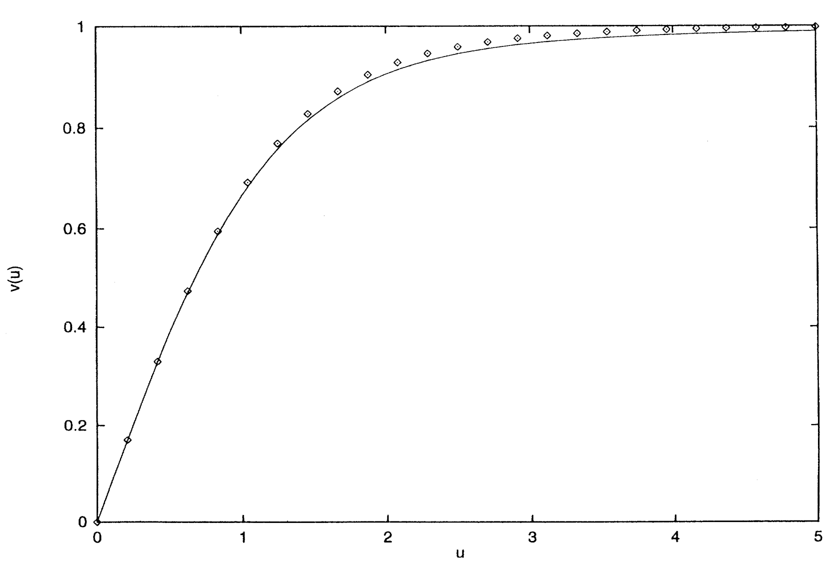

FIG. 2. Plot of disclination profile $v(u)$. Line was obtained by numerical integration of Eq. (7), diamonds represent Eq. (9).

where we have neglected the quantity $2 - 2v^2$, which is also negligible except in the vicinity of the disclination. Equation (16) is Bessel's equation, and the solution with $g(0)$ finite and $g(R/R_d) = 0$ is

$$g_{mn}(u) = \sqrt{2}\frac{R_d}{R}\frac{J_{\bar{m}}(\xi_{\bar{m}n}u)}{J_{\bar{m}+1}(\xi_{\bar{m}n})}, \tag{17}$$

where

$$\bar{m} = \left(1 + \frac{m^2}{\gamma}\right)^{1/2}, \tag{18}$$

$$\mathcal{E}_{mn} = \gamma\xi^2_{\bar{m}n}\left(\frac{R_d}{R}\right)^2, \tag{19}$$

n is a positive integer (the radial quantum number), $\xi_{\bar{m}n}$ is the nth root of the Bessel function of order $\bar{m}$, and we have adopted the normalization

$$1 = \int du\, u\, (f^2 + g^2). \tag{20}$$

The corresponding normal modes

$$\Lambda_{\ell mn} = i g_{mn}(u)\, \hat{\phi}\, e^{im\phi}\, e^{ik_\ell z} \tag{21}$$

represent fluctuations that have only an azimuthal component. Radial fluctuations are either confined to the neighborhood of the disclination or have large eigenvalues, and are neglected in this approximation.

The normal mode expansions for the real quantities $\mathbf{p}_1$ and F_2 are

$$\mathbf{p}_1 = \sum_{\ell mn} i A_{\ell mn}\Lambda_{\ell mn}, \tag{22}$$

and

$$F_2 = \pi K L \sum_{\ell m n} |A_{\ell mn}|^2 \left[\mathcal{E}_{mn} + (k_\ell R_d)^2\right], \tag{23}$$

where $A_{\ell,m,n} = A^*_{-\ell,-m,n}$ are the expansion coefficients, ℓ and m are summed over all integers, and n is summed over positive integers. These are the normal mode expansions we set out to find.

III. CORRELATION FUNCTION

Since our ultimate goal is to calculate the scattering and attenuation of the fiber, we need to relate thermal fluctuations to fluctuations in the dielectric tensor, which cause scattering. The Sm-C director may be written

$$\hat{\mathbf{n}} = p_y\,\hat{\mathbf{x}} - p_x\,\hat{\mathbf{y}} + \hat{\mathbf{z}}, \tag{24}$$

where we assume $p \ll 1$ (i.e., the Sm-C tilt angle is small) and neglect terms of order p^2. The dielectric tensor is [21–23]

$$\varepsilon_{ij} = \varepsilon_\perp\delta_{ij} + \varepsilon_a n_i n_j, \tag{25}$$

where $\varepsilon_\parallel$ and $\varepsilon_\perp$ are the dielectric constants parallel and perpendicular to the director, and $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp$. To first order in p_1 we find

$$\delta\varepsilon = \varepsilon_a \begin{pmatrix} 0 & 0 & p_{1y} \\ 0 & 0 & -p_{1x} \\ p_{1y} & -p_{1x} & 0 \end{pmatrix}, \tag{26}$$

where $\delta\varepsilon = \varepsilon - \varepsilon_0$ and ε_0 is the dielectric tensor at zero temperature.

When calculating the light scattering in Sec. IV we will need the spatial correlation function

$$S(\mathbf{r},\mathbf{r}') = \langle\psi^*(\mathbf{r})\,\psi(\mathbf{r}')\rangle = \frac{\int \mathcal{D}\psi\, e^{-F_2/T}\,\psi^*(\mathbf{r})\,\psi(\mathbf{r}')}{\int \mathcal{D}\psi\, e^{-F_2/T}}, \tag{27}$$

where

$$\begin{aligned}\psi(\mathbf{r}) &= \varepsilon_a\, p_{1y}(\mathbf{r}) \\ &= -\sqrt{2}\varepsilon_a (R_d/R)\,\cos\phi \\ &\quad\times \sum_{\ell mn} A_{\ell mn}\frac{J_{\bar{m}}(\xi_{\bar{m}n}u)\, e^{im\phi}\, e^{ik_\ell z}}{J_{\bar{m}+1}(\xi_{\bar{m}n})},\end{aligned} \tag{28}$$

$\langle\,\rangle$ denotes a thermal average, T is temperature (in energy units), and the functional integrals are to be evaluated by varying the real and imaginary parts of $A_{\ell mn}$ from $-\infty$ to $+\infty$ subject to the constraint $A_{\ell,m,n} = A^*_{-\ell,-m,n}$. Substituting from (19), (23), and (28), and evaluating the functional integrals, then taking the limit $L \to \infty$, gives

$$\begin{aligned}S(\mathbf{r},\mathbf{r}') = {}& \frac{4\varepsilon_a^2 T\,\cos\phi\,\cos\phi'}{\pi\sqrt{\gamma}KR} \\ &\times \sum_{mn}\frac{J_{\bar{m}}(\xi_{\bar{m}n}\rho/R)\,J_{\bar{m}}(\xi_{\bar{m}n}\rho'/R)}{\xi_{\bar{m}n}\,J^2_{\bar{m}+1}(\xi_{\bar{m}n})} \\ &\times \cos[m(\phi-\phi')]\,e^{-\sqrt{\gamma}\xi_{\bar{m}n}|z-z'|/R}.\end{aligned} \tag{29}$$

Because of the exponential factor, if γ is large, correlations are confined to thin transverse sections of the fiber. (Recall that $\gamma = K_1/K$ is of the order of the degree of polymerization.)

IV. SCATTERING AND ATTENUATION

In this section we estimate the scattering distribution and attenuation of an optical fiber with a Sm-C LCP core. We make several assumptions to simplify the analysis. First, we assume the Sm-C tilt angle is small ($\theta_0 \ll 1$), so that the optical axis of the birefringent core is along $\hat{\mathbf{z}}$, and we assume the refractive indices of the core ($n_\parallel = \varepsilon_\parallel^{1/2}$, $n_\perp = \varepsilon_\perp^{1/2}$) are spatially homogeneous. Such fibers are known as *step-index* fibers. Second, we assume the birefringence is small ($|\varepsilon_a| \ll 1$, $|n_\parallel - n_\perp| \ll 1$). Third, we assume the cladding has an isotropic refractive index n_{cl}, that it fills all space outside the core, and that it has negligible attenuation. Fourth, we assume the refractive indices of the core and cladding are nearly the same ($0 < n_\perp - n_{\text{cl}} \ll n_{\text{cl}}$). Fibers that satisfy this condition are said to be *weakly guiding*. Finally, we restrict our attention to the lowest-order linearly polarized guided mode (LP_{01}), which has no azimuthal variation and no radial nodes. Fibers designed to support only this mode are said to be *single mode.* A weakly guiding fiber is single mode if $W < 2.405$, where

$$W = kR\sqrt{n_\perp^2 - n_{\rm cl}^2}\,, \tag{30}$$

and k is the vacuum wave number of the light [24]. Typical communication fibers are step index, weakly guiding, and single mode.

Our first task is to write down the electric field for the guided light propagating in the fiber core. The electric field propagating within the core of a step-index, weakly guiding, single-mode fiber is [24,25]

$$\mathbf{E}_0(\mathbf{r}) = \hat{\mathbf{x}} E\, J_0(\xi\rho/R)\, e^{i\kappa_0 z}\,, \tag{31}$$

where

$$\xi(W) = \frac{(1+\sqrt{2})W}{1+(4+W^4)^{1/4}}\,, \tag{32}$$

E is the on-axis field amplitude, $\kappa_0 = n_\perp k$ is the propagation wave number, and the time dependence $e^{-i\omega t}$ is understood. The attenuation of light propagating along a fiber with cylindrical symmetry cannot depend on its direction of polarization, so there is no loss of generality in assuming that $\mathbf{E}_0$ is polarized along $\hat{\mathbf{x}}$.

Now that we know the guided field $\mathbf{E}_0(\mathbf{r})$ and the correlation function $S(\mathbf{r},\mathbf{r}')$ within the core, we are in a position to estimate the attenuation. We will do the calculation in two different ways. The first approach is simple and physically transparent, but involves the (apparently severe) approximation that all three refractive indices ($n_\parallel$, $n_\perp$, and $n_{\rm cl}$) are identical. We refer to this calculation as the zeroth-order approximation. The second approach only requires that the core birefringence and core-cladding refractive index differences are small. We refer to this calculation as the first-order approximation.

A. Zeroth-order approximation

In this section we neglect core birefringence and core-cladding refractive index differences entirely, i.e., we take

$$n_\parallel = n_\perp = n_{\rm cl} = n\,. \tag{33}$$

We view this as a zeroth-order approximation. The calculation is based on dipole scattering in the far-field limit [26].

The induced electric polarization in the core due to the guided light is [23]

$$\mathbf{P}(\mathbf{r}) = \left\{\frac{\varepsilon(\mathbf{r})-1}{4\pi}\right\}\cdot\mathbf{E}_0(\mathbf{r})\,, \tag{34}$$

where the quantity in braces is the electric susceptibility tensor, and we adopt Gaussian units. The induced polarization at $\mathbf{r}$ radiates an electromagnetic field. The radiated electric field at $\mathbf{r}'$ per unit volume of induced polarization at $\mathbf{r}$ is [23]

$$\mathbf{E}(\mathbf{r},\mathbf{r}') = \frac{k^2\, e^{i\kappa_0 s}\, \{1-\hat{\mathbf{s}}\hat{\mathbf{s}}\}\cdot\mathbf{P}(\mathbf{r})}{s}\,, \tag{35}$$

where $\mathbf{s} = \mathbf{r}'-\mathbf{r}$, and the tensor in braces projects out the component of $\mathbf{P}$ orthogonal to $\mathbf{s}$. The scattered electric field $\mathbf{E}_s(\mathbf{r}')$ is obtained by integrating all the contributions $\mathbf{E}(\mathbf{r},\mathbf{r}')$ over the fiber core:

$$\mathbf{E}_s(\mathbf{r}') = \int d\mathbf{r}\, \mathbf{E}(\mathbf{r},\mathbf{r}')\,. \tag{36}$$

To evaluate radiative losses, it is sufficient to calculate $\mathbf{E}_s$ in the far-field limit ($s\to\infty$), with the approximations

$$\frac{e^{i\kappa_0 s}}{s} \simeq \frac{e^{i\kappa_0 \hat{\mathbf{r}}'\cdot(\mathbf{r}'-\mathbf{r})}}{r'}\,, \qquad \hat{\mathbf{s}} \simeq \hat{\mathbf{r}}'\,. \tag{37}$$

We also replace $\varepsilon - 1$ with $\delta\varepsilon$, since only fluctuations in the susceptibility contribute to scattering. Then

$$\mathbf{E}_s(\mathbf{s}) \simeq \frac{E\,\cos\theta_s\, e^{i\kappa_0 s}\, \alpha(\hat{\mathbf{s}})\,\hat{\mathbf{e}}}{s}\,, \tag{38}$$

where

$$\alpha(\hat{\mathbf{s}}) = \frac{k^2}{4\pi}\int d\mathbf{r}\,\psi(\mathbf{r})\, J_0(\xi\rho/R)\, e^{i\kappa_0\mathbf{r}\cdot(\hat{\mathbf{z}}-\hat{\mathbf{s}})}\,, \tag{39}$$

$$\hat{\mathbf{e}} = -\hat{\mathbf{x}}\,\cos\theta_s\,\cos\phi_s - \hat{\mathbf{y}}\,\cos\theta_s\,\sin\phi_s + \hat{\mathbf{z}}\,\sin\theta_s\,, \tag{40}$$

$\psi(\mathbf{r})$ is given by (28), and θ_s and ϕ_s are the polar and azimuthal angles of $\hat{\mathbf{s}}$. Far from the fiber core, $\mathbf{E}_s(\mathbf{s})$ is a plane wave radiating along $\hat{\mathbf{s}}$ and polarized along $\hat{\mathbf{e}}$.

The total scattered power is [26]

$$\begin{aligned}\mathcal{P}_s &= \frac{cn}{8\pi}\int d\Omega_s\, s^2\,\left\langle|\mathbf{E}_s(\mathbf{s})|^2\right\rangle \\ &= \frac{cnE^2}{8\pi}\int d\Omega_s\,\sin^2\theta_s\,\left\langle|\alpha(\hat{\mathbf{s}})|^2\right\rangle\,,\end{aligned} \tag{41}$$

where

$$\begin{aligned}\left\langle|\alpha(\hat{\mathbf{s}})|^2\right\rangle &= \frac{k^4}{16\pi^2}\int d\mathbf{r}\int d\mathbf{r}'\, S(\mathbf{r},\mathbf{r}') \\ &\quad\times J_0(\xi\rho/R)\, J_0(\xi\rho'/R)\, e^{i\kappa_0(\hat{\mathbf{z}}-\hat{\mathbf{s}})\cdot(\mathbf{r}'-\mathbf{r})}\,,\end{aligned}$$

$S(\mathbf{r},\mathbf{r}')$ is given by (29), and $d\Omega_s = \sin\theta_s\, d\theta_s\, d\phi_s$ is the differential solid angle associated with $\hat{\mathbf{s}}$. The integrations over z, z', ϕ_s, ϕ, and ϕ' can be done analytically, giving

$$\begin{aligned}\mathcal{P}_s = \frac{\varepsilon_a^2 T E^2 cnL(kR)^4}{8\pi K}\sum_{mn}\int_0^\pi d\theta_s\,\sin^3\theta_s \\ \times\left\{A^+_{mn}(\kappa_0 R\sin\theta_s,\xi) + A^-_{mn}(\kappa_0 R\sin\theta_s,\xi)\right\} \\ \times\Omega_{mn}(1-\cos\theta_s)\,,\end{aligned} \tag{42}$$

where

$$A^\pm_{mn}(\nu,\xi) = \left(\frac{\int_0^1 dx\, x\, J_{|m\pm1|}(\nu x)\, J_0(\xi x)\, J_{\bar m}(\xi_{\bar m n} x)}{J_{\bar m+1}(\xi_{\bar m n})}\right)^2\,,$$

$$\Omega_{mn}(\nu) = \frac{1}{\gamma\xi^2_{\bar m n} + (\kappa_0 R\nu)^2}\,.$$

The way scattering depends on the polar angle θ_s is evident from the θ_s integrand: the factor $\sin^3\theta_s$ strongly suppresses grazing-angle scattering, while the factor $\Omega_{mn}(1-\cos\theta_s)$—which stems from the integrations over z and z'—slightly enhances it. The integers m and n are the azimuthal and radial quantum numbers, and each term of the double summation indicates the extent to which the corresponding scattering mode is excited. The quantity $\hbar m$ is the z component of the angular momentum of photons scattered into azimuthal mode m.

The remaining sums and integrals are analytically intractable, and numerical evaluation is complicated by their dependence on several parameters (γ, ξ, and $\kappa_0 R$). To simplify matters, assume the degree of polymerization is large, so that $\gamma \gg 1$. Then $\bar{m} \simeq 1$ and $\Omega_{mn} \simeq \gamma^{-1}\xi_{1n}^{-2}$. Also, for a single-mode fiber, $J_0(\xi x) \simeq 1$ over the interval $0 \leq x \leq 1$, and $J_1(\xi) \simeq 0$. With these approximations, the expression for the total scattered power becomes

$$\mathcal{P}_s \simeq \frac{4\varepsilon_a^2 T L (kR)^4}{\pi\gamma K R^2} B(\kappa_0 R)\,\mathcal{P}_{co}\,, \tag{43}$$

where

$$\mathcal{P}_{co} \simeq \frac{cnE^2R^2}{8}$$

is the light power carried in the core [24], and

$$B(\kappa_0 R) = \sum_{mn} \int_0^{\pi/2} d\theta_s \, \sin^3\theta_s \times \left(\frac{\int_0^1 dx\, x\, J_{|m|}(\kappa_0 R \sin\theta_s\, x)\, J_1(\xi_{1n} x)}{\xi_{1n}\, J_2(\xi_{1n})} \right)^2 .$$

The function $B(x)$ was evaluated numerically, and is plotted in Fig. 3.

The $1/e$ attenuation length of the fiber is

$$\mu = \left(\frac{\mathcal{P}_s}{L\mathcal{P}}\right)^{-1} \simeq \frac{\pi R^2 \gamma K}{4\varepsilon_a^2 T\, B(\kappa_0 R)\, (kR)^4} \left(\frac{\mathcal{P}_{co}}{\mathcal{P}}\right)^{-1}, \tag{44}$$

where $\mathcal{P}$ is the total (core plus cladding) power carried in the fiber. This is the zeroth-order approximation for the fiber attenuation.

B. First-order approximation

In this section we again calculate the fiber scattering distribution and attenuation, assuming the core birefringence and core-cladding refractive index differences are small but non-zero. We regard this as a first-order approximation. We use time-dependent perturbation theory in its most familiar form—the golden rule of quantum mechanics [27]—and, when convenient, borrow notation and terminology from quantum mechanics (*bra-ket* notation, the word "wave function," etc.). However, our calculation is, in truth, classical—Planck's constant does not appear in the final results.

Let $|\mathbf{E}_0\rangle$ represent the (unnormalized) wave function of a photon in the LP_{01} bound (guided) mode. A photon that scatters out of the fiber can be thought of as making a transition from the bound mode to an unbound mode. Let $|\mathbf{E}_s\rangle$ represent the (unnormalized) wave function of a photon in an unbound (scattered) mode. Then the number of photons per unit time per unit scattering angle scattered from the bound mode into an unbound mode with scattering angle θ_s is given by the golden rule [27]:

$$\frac{d\Omega}{d\theta_s}(\theta_s) = \frac{2\pi}{\hbar} \sum \mathcal{G} \frac{\langle |\langle \mathbf{E}_s | \hbar\omega\, \delta\varepsilon | \mathbf{E}_0 \rangle|^2 \rangle}{\langle \mathbf{E}_s | \mathbf{D}_s \rangle \langle \mathbf{E}_0 | \mathbf{D}_0 \rangle}\,, \tag{45}$$

where $\hbar$ is Planck's constant, ω is the angular frequency

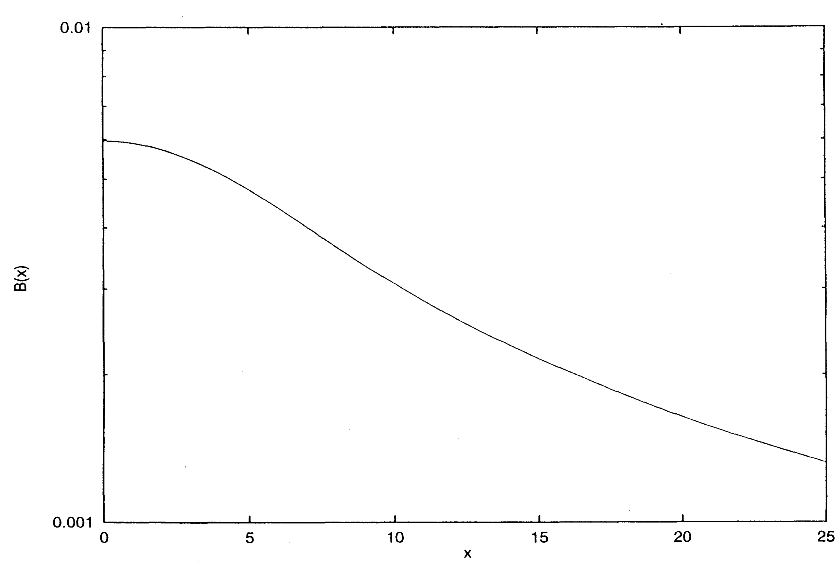

FIG. 3. Plot of scattering function $B(x)$.

of the light, $\mathbf{D}_0$ and $\mathbf{D}_s$ are electric displacement vectors, $\mathcal{G}$ represents the density of final states (i.e., the number of final states per unit energy per unit scattering angle), and the summation is over all unbound modes with scattering angle θ_s that conserve energy (i.e., with vacuum wave number k). The inner product $\langle \mathbf{E}_s|\hbar\omega\,\delta\varepsilon|\mathbf{E}_0\rangle$ represents the integral

$$\int d\mathbf{r}\,\hbar\omega\,\mathbf{E}_s^*(\mathbf{r})\cdot\delta\varepsilon\cdot\mathbf{E}_0(\mathbf{r})\,,$$

and $\langle \mathbf{E}_s|\mathbf{D}_s\rangle$ and $\langle \mathbf{E}_0|\mathbf{D}_0\rangle$ are defined analogously. The quantity $\hbar\omega\,\delta\varepsilon$ is regarded as the perturbation that couples bound and unbound modes. The total scattered power is then

$$\mathcal{P}_s = \left\{\frac{\langle \mathbf{E}_0|\mathbf{D}_0\rangle}{16\pi}\right\}\Omega \qquad (46)$$

$$= \frac{\hbar c^2 k^2}{8}\int_0^\pi d\theta_s \sum \mathcal{G}\frac{\langle|\langle \mathbf{E}_s|\delta\varepsilon|\mathbf{E}_0\rangle|^2\rangle}{\langle \mathbf{E}_s|\mathbf{D}_s\rangle}\,,$$

where c is the vacuum speed of light, and the quantity in braces is the total energy [26] in the bound mode.

The bound mode $|\mathbf{E}_0\rangle$ is given (within the core) by (31). Unbound modes are discussed in Appendix A. Each component of $|\mathbf{E}_s\rangle$ has the functional form

$$h(\rho)\,e^{im_s\phi}\,e^{i\kappa z}\,,$$

where m_s (an integer) is the azimuthal quantum number, $\kappa = n_{\rm cl}k\,\cos\theta_s$ is the propagation wave number, and the radial functions $h(\rho)$ differ in the core and cladding, and from component to component. An unbound mode is fully specified by the three parameters k, θ_s, and m_s. Note that k is constrained by energy conservation to be the same for the scattered and guided light, and that the θ_s dependence in the expression for $\mathcal{P}_s$ is explicitly exhibited, so that the summation over final states amounts to a summation over all integers m_s.

In order to calculate the density of states $\mathcal{G}$ and the normalization integral $\langle \mathbf{E}_s|\mathbf{D}_s\rangle$, it is convenient to introduce "box normalization," i.e., to imagine the fiber concentrically enclosed in a cylindrical universe of (arbitrarily large) radius R_∞. In the two-dimensional box of length L and width R_∞, the number of states per unit scattering angle in the interval δk at k is

$$\frac{LR_\infty n_{\rm cl}^2 k\,\delta k}{(2\pi)^2}\,.$$

The energy associated with δk is $\hbar c\,\delta k$, so the number of states per unit energy per unit scattering angle is

$$\mathcal{G} = \frac{LR_\infty n_{\rm cl}^2 k}{(2\pi)^2\hbar c}\,. \qquad (47)$$

The calculation of $\langle \mathbf{E}_s|\mathbf{D}_s\rangle$ is outlined in Appendix A. If the core birefringence is small and the fiber is weakly guiding, then

$$\langle \mathbf{E}_s|\mathbf{D}_s\rangle = \frac{2LR_\infty n_{\rm cl}E_s^2}{k\,\sin^3\theta_s}\,, \qquad (48)$$

where E_s characterizes the strength of the scattered field. (The quantity E_s will not appear in our final results, and could be set equal to one—as is done in Appendix A. Here we retain E_s so that the dimensional consistency of our equations is manifest.) The core constitutes only an infinitesimal portion of the box, and this is why the refractive indices of the core do not appear in (47) and (48). Equation (46) for the total scattered power can now be written as

$$\mathcal{P}_s = \frac{cn_{\rm cl}k^4}{64\pi^2E_s^2}\int_0^\pi d\theta_s\,\sin^3\theta_s \sum_{m_s}\langle|\langle \mathbf{E}_s|\delta\varepsilon|\mathbf{E}_0\rangle|^2\rangle\,. \quad (49)$$

The quantity $\delta\varepsilon$ is zero outside the core, and the product $\delta\varepsilon\cdot\mathbf{E}_0$ has only a z component. So only the z component of $\mathbf{E}_s$ within the core is required to evaluate (49). From Appendix A,

$$\hat{\mathbf{z}}\cdot\mathbf{E}_s(\mathbf{r}) = E_s\,J_{m_s}(\eta U\rho/R)\,e^{im_s\phi}\,e^{i\kappa z}\,, \qquad (50)$$

where $\eta = n_\parallel/n_\perp$ is a birefringence parameter, and

$$U = R\sqrt{n_\perp^2k^2-\kappa^2}\,. \qquad (51)$$

Then (31) and (50) imply

$$\sum_{m_s}\langle|\langle \mathbf{E}_s|\delta\varepsilon|\mathbf{E}_0\rangle|^2\rangle = E^2E_s^2\sum_{m_s}\int d\mathbf{r}\int d\mathbf{r}'\,S(\mathbf{r},\mathbf{r}')$$
$$\times J_0(\xi\rho/R)\,J_0(\xi\rho'/R)$$
$$\times J_{m_s}(\eta U\rho/R)\,J_{m_s}(\eta U\rho'/R)$$
$$\times e^{im_s(\phi-\phi')}\,e^{i(\kappa-\kappa_0)(z-z')}\,, \qquad (52)$$

where $S(\mathbf{r},\mathbf{r}')$ is given by (29), and the integration is over the fiber core. The integrations over z, z', ϕ, and ϕ' can be done analytically, giving

$$\mathcal{P}_s = \frac{\varepsilon_a^2TE^2cn_{\rm cl}L(kR)^4}{8\pi\gamma K}\sum_{mn}\int_0^\pi d\theta_s\,\sin^3\theta_s$$
$$\times[A_{mn}^+(\eta U,\xi)+A_{mn}^-(\eta U,\xi)]$$
$$\times\Omega_{mn}\left(1-\frac{n_{\rm cl}}{n_\perp}\cos\theta_s\right)\,, \qquad (53)$$

where $A_{mn}^\pm(\nu,\xi)$ and $\Omega_{mn}(\nu)$ are defined in (42). To lowest order in $n_\perp - n_\parallel$ and $n_\perp - n_{\rm cl}$,

$$\eta U \simeq \kappa_0R\,\sin\theta_s\left(1-\frac{n_\perp-n_\parallel}{n_\perp}+\frac{n_\perp-n_{cl}}{n_\perp}\cot^2\theta_s\right)\,. \qquad (54)$$

This first-order result for $\mathcal{P}_s$ is to be compared with the zeroth-order result (42). In the limit of no birefringence and no core-cladding refractive index differences, the two expressions are identical. If $n_\perp - n_{\rm cl} > 0$, then the $\cot^2\theta_s$ term in (54) diverges as $\theta_s \to 0$, which drives the $A_{mn}^\pm$ terms to zero. So grazing-angle scattering is suppressed even more strongly than $\sin^3\theta_s$. This appears to be the most pronounced difference between the first-order and zeroth-order results.

We emphasize that (53) is not a true first-order approximation in the small quantities $n_\perp - n_\parallel$ and $n_\perp - n_{\rm cl}$. There are two reasons. The first is the way we handled the normalization factor $\langle \mathbf{E}_s | \mathbf{D}_s \rangle$. In going from (A19) to (A20) we made a zeroth-order approximation, neglecting the birefringence and core-cladding refractive index differences entirely. A true first-order treatment would introduce additional factors into the expression for $\mathcal{P}_s$—factors that depend on θ_s and m, and may have first-order deviations from unity. Such a treatment is difficult due to the complexity of Eqs. (A10). This complexity is due to interference patterns in the scattered fields, which, from the point of view of ray optics, stem from multiple reflections from the core-cladding interface. The second reason the approximation is not truly first order is that we discarded the ITE modes (incident transverse electric; see Appendix A). One could calculate a separate $\mathcal{P}_s$ for the ITE modes, and it may contain first-order terms. In the overall spirit of the calculation (recall that we have already assumed $K \sim K_2 \sim K_3$ and $\theta_0 \ll 1$), we retained only the first-order terms that were simplest to treat—those which stem from the interaction of the bound and ITM (incident transverse magnetic) scattered modes. But, clearly, our approach could be extended to ITE modes, and to any desired order of approximation, at least in principle.

V. DISCUSSION

The zeroth-order assumption $n_\parallel = n_\perp = n_{\rm cl}$ ignores the effects of refraction at the core-cladding interface. For a weakly guiding fiber it is appropriate to neglect refraction—except at grazing angles, when $\theta_s \ll 1$ [24]. But the factor $\sin^3\theta_s$ in (42) suppresses grazing-angle contributions to the scattered power, so we expect the zeroth-order approximation to be rather good. This is confirmed by the first-order result, which reveals that grazing-angle scattering is suppressed even more strongly than $\sin^3\theta_s$.

The form of the expression for attenuation length (44) admits physical interpretation. The ubiquitous factor k^4 is Rayleigh's law [26]. The ratio $\mathcal{P}_{co}/\mathcal{P}$ is the fraction of total guided power carried in the core. As W increases from 0 to 2.405, this ratio increases from 0 to about 0.8 [24]. The fact that μ is inversely proportional to the fraction of power in the core is reasonable, since, in our fiber model, only the power in the core is susceptible to scattering. The fact that μ is directly proportional to γ is also reasonable, since $\gamma = K_1/K$ characterizes the elastic stiffness of the system, and stiff systems resist thermal fluctuations. Since γ goes as the degree of polymerization, so does the attenuation length.

We can make an order-of-magnitude estimate of the $1/e$ attenuation length μ. Let $R \sim 10^{-6}$ m, $K \sim 10^{-11}$ J m^{-1} [23], $\varepsilon_a \sim 0.1$, $n \sim 1.5$, and $T \sim 4 \times 10^{-21}$ J, and take $\kappa_0 R \sim 2\pi$. Then $kR = \kappa_0 R/n \sim 4$, $B(\kappa_0 R) \sim 0.004$ (see Fig. 3), and (43) gives

$$\mu \sim 0.2\,\gamma \left(\frac{\mathcal{P}_{co}}{\mathcal{P}}\right)^{-1} \text{m}\,. \tag{55}$$

If the degree of polymerization is large ($\gamma \gg 1$), and if most of the light is carried in the core ($\mathcal{P}_{co}/\mathcal{P} \sim 1$), then the $1/e$ attenuation length is of the order of 1 m or more. This is an encouraging result, at least from the standpoint of using fibers with Sm-C LCP cores for special-purpose applications.

There are, of course, other scattering mechanisms in optical fibers besides thermal fluctuations of the core refractive index. Another fundamental loss mechanism is that due to thermal fluctuations of the core-cladding interface. This mechanism is briefly discussed in Appendix B, where we estimate the corresponding $1/e$ attenuation length to be much greater than 10^4 m. So, this loss mechanism is negligible compared to thermal fluctuations of the core refractive index.

Material inhomogeneities, core-cladding interface irregularities, bending, and so forth, all contribute to losses. But, in principle, these losses can be made negligible. Scattering due to thermal fluctuations is fundamental, and represents the dominant loss mechanism in a well-fabricated fiber.

ACKNOWLEDGMENTS

This work was supported by the NSF Material Research Group program under Grant No. DMR-91-22227.

APPENDIX A: UNBOUND MODES

The unbound modes of a step-index fiber are derived by Snyder [28] for the case of a nonbirefringent core. Here we extend his analysis to the case of a birefringent core. The reader is referred to [28] and [29] for additional details.

1. Fields in the core

We write the electric and magnetic fields in the form

$$\begin{aligned} &\mathbf{E}(\rho,\phi)\, e^{i\kappa z - i\omega t}\,, \\ &\mathbf{H}(\rho,\phi)\, e^{i\kappa z - i\omega t}\,, \end{aligned} \tag{A1}$$

and solve Maxwell's equations in cylindrical coordinates. Due to the cylindrical symmetry of the fiber (translation and rotation invariance with respect to $\hat{\mathbf{z}}$), the z components of $\mathbf{E}$ and $\mathbf{H}$ satisfy ordinary wave equations. Substituting from (A1) gives, inside the core,

$$\begin{aligned} &\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\eta^2 U^2}{R^2} E_z = 0\,, \\ &\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{U^2}{R^2} H_z = 0\,, \end{aligned} \tag{A2}$$

where $k = \omega/c$ is the vacuum wave number of the light, $n_\parallel$ and $n_\perp$ are the longitudinal and transverse refractive indices of the core, $\eta = n_\parallel/n_\perp$ is a birefringence parameter, and

$$U = R\sqrt{n_\perp^2 k^2 - \kappa^2}\,. \tag{A3}$$

The solutions of (A2) that remain finite as $\rho \to 0$ are

$$E_z(\rho,\phi) = a_1\, J_m(\eta U\rho/R)\, e^{im\phi}\,, \qquad \text{(A4)}$$
$$H_z(\rho,\phi) = a_2\, J_m(U\rho/R)\, e^{im\phi}\,,$$

where J_m is the Bessel function of the first kind, m (an integer) is the azimuthal quantum number (denoted m_s in the text), and a_1 and a_2 are constants. The remaining field components in the core can be calculated from E_z and H_z [29]:

$$\begin{aligned}
E_\rho &= i\frac{R^2}{U^2}\left(\kappa\frac{\partial E_z}{\partial\rho} + \frac{k}{\rho}\frac{\partial H_z}{\partial\phi}\right)\\
&= ia_1\frac{\eta\kappa R}{U}J_m'(\eta U\rho/R)\,e^{im\phi}\\
&\quad -a_2\frac{kR^2m}{U^2\rho}J_m(U\rho/R)\,e^{im\phi}\,,\\
H_\rho &= i\frac{R^2}{U^2}\left(\kappa\frac{\partial H_z}{\partial\rho} - \frac{kn_\perp^2}{\rho}\frac{\partial E_z}{\partial\phi}\right)\\
&= ia_2\frac{\kappa R}{U}J_m'(U\rho/R)\,e^{im\phi}\\
&\quad +a_1\frac{kR^2n_\perp^2 m}{U^2\rho}J_m(\eta U\rho/R)\,e^{im\phi}\,,\\
E_\phi &= i\frac{R^2}{U^2}\left(\frac{\kappa}{\rho}\frac{\partial E_z}{\partial\phi} - k\frac{\partial H_z}{\partial\rho}\right)\\
&= -ia_2\frac{kR}{U}J_m'(U\rho/R)\,e^{im\phi}\\
&\quad -a_1\frac{\kappa R^2 m}{U^2\rho}J_m(\eta U\rho/R)\,e^{im\phi}\,,\\
H_\phi &= i\frac{R^2}{U^2}\left(\frac{\kappa}{\rho}\frac{\partial H_z}{\partial\phi} + kn_\perp^2\frac{\partial E_z}{\partial\rho}\right)\\
&= ia_1\frac{\eta kRn_\perp^2}{U}J_m'(\eta U\rho/R)\,e^{im\phi}\\
&\quad -a_2\frac{\kappa R^2 m}{U^2\rho}J_m(U\rho/R)\,e^{im\phi}\,. \qquad \text{(A5)}
\end{aligned}$$

2. Fields in the cladding

In the cladding, the components E_z and H_z are given by

$$\frac{\partial^2 E_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{V^2}{R^2}E_z = 0\,, \qquad \text{(A6)}$$
$$\frac{\partial^2 H_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\phi^2} + \frac{V^2}{R^2}H_z = 0\,,$$

where $n_{\rm cl}$ is the refractive index of the cladding, and

$$V = R\sqrt{n_{\rm cl}^2k^2 - \kappa^2}\,. \qquad \text{(A7)}$$

The solutions of (A6) are

$$\begin{aligned}
E_z(\rho,\phi) &= a_3\, J_m(V\rho/R)\, e^{im\phi} \qquad \text{(A8)}\\
&\quad + a_4\, H_m^{(1)}(V\rho/R)\, e^{im\phi}\,,\\
H_z(\rho,\phi) &= a_5\, J_m(V\rho/R)\, e^{im\phi}\\
&\quad + a_6\, H_m^{(1)}(V\rho/R)\, e^{im\phi}\,,
\end{aligned}$$

where $H_m^{(1)}$ is the Hankel function of the first kind, and a_3, a_4, a_5, and a_6 are constants. The remaining field components in the cladding are

$$\begin{aligned}
E_\rho &= i\frac{R^2}{V^2}\left(\kappa\frac{\partial E_z}{\partial\rho} + \frac{k}{\rho}\frac{\partial H_z}{\partial\phi}\right)\\
&= ia_3\frac{\kappa R}{V}J_m'(V\rho/R)\,e^{im\phi}\\
&\quad -a_5\frac{kR^2m}{V^2\rho}J_m(V\rho/R)\,e^{im\phi}\\
&\quad +ia_4\frac{\kappa R}{V}H_m^{(1)\prime}(V\rho/R)\,e^{im\phi}\\
&\quad -a_6\frac{kR^2m}{V^2\rho}H_m^{(1)}(V\rho/R)\,e^{im\phi}\,,\\
H_\rho &= i\frac{R^2}{V^2}\left(\kappa\frac{\partial H_z}{\partial\rho} - \frac{kn_{\rm cl}^2}{\rho}\frac{\partial E_z}{\partial\phi}\right)\\
&= ia_5\frac{\kappa R}{V}J_m'(V\rho/R)\,e^{im\phi}\\
&\quad +a_3\frac{kR^2n_{\rm cl}^2m}{V^2\rho}J_m(V\rho/R)\,e^{im\phi}\\
&\quad +ia_6\frac{\kappa R}{V}H_m^{(1)\prime}(V\rho/R)\,e^{im\phi}\\
&\quad +a_4\frac{kR^2n_{\rm cl}^2m}{V^2\rho}H_m^{(1)}(V\rho/R)\,e^{im\phi}\,,\\
E_\phi &= i\frac{R^2}{V^2}\left(\frac{\kappa}{\rho}\frac{\partial E_z}{\partial\phi} - k\frac{\partial H_z}{\partial\rho}\right)\\
&= -ia_5\frac{kR}{V}J_m'(V\rho/R)\,e^{im\phi}\\
&\quad -a_3\frac{\kappa R^2m}{V^2\rho}J_m(V\rho/R)\,e^{im\phi}\\
&\quad -ia_6\frac{kR}{V}H_m^{(1)\prime}(V\rho/R)\,e^{im\phi}\\
&\quad -a_4\frac{\kappa R^2m}{V^2\rho}H_m^{(1)}(V\rho/R)\,e^{im\phi}\,,\\
H_\phi &= i\frac{R^2}{V^2}\left(\frac{\kappa}{\rho}\frac{\partial H_z}{\partial\phi} + kn_{\rm cl}^2\frac{\partial E_z}{\partial\rho}\right)\\
&= ia_3\frac{kRn_{\rm cl}^2}{V}J_m'(V\rho/R)\,e^{im\phi}\\
&\quad -a_5\frac{\kappa R^2m}{V^2\rho}J_m(V\rho/R)\,e^{im\phi}\\
&\quad +ia_4\frac{kRn_{\rm cl}^2}{V}H_m^{(1)\prime}(V\rho/R)\,e^{im\phi}\\
&\quad -a_6\frac{\kappa R^2m}{V^2\rho}H_m^{(1)}(V\rho/R)\,e^{im\phi}\,. \qquad \text{(A9)}
\end{aligned}$$

3. Boundary conditions

At the core-cladding interface ($\rho = R$) we impose the usual boundary conditions: the tangential components of **E** and **H**, and the normal components of **D** and **B** (the electric displacement and magnetic induction), must be continuous. Two of these six conditions turn out to be redundant; it is sufficient to impose continuity only on the four tangential components E_z, H_z, E_ϕ, and H_ϕ. Thus we have four boundary conditions and six unknown constants $(a_1, \ldots, a_6)$. One of these constants may be chosen arbitrarily, to set the overall intensity of the electromagnetic field. So we actually have five independent constants, and four boundary conditions. One more condition is needed.

Snyder provides this extra condition in the following way. He divides the unbound modes into two classes: ITM and ITE. The ITM (incident transverse magnetic) modes have $a_5 = 0$ as the extra condition, and the ITE (incident transverse electric) modes have $a_3 = 0$ as the extra condition. See [28] for a detailed discussion of ITM and ITE modes, and motivation of the terminology.

4. ITM modes

Combining the boundary conditions with the extra condition $a_5 = 0$ gives a linear set of five equations in five unknowns, with the solution

$$
\begin{aligned}
\frac{a_2}{a_1} &= i\frac{\kappa m\Theta_1}{k\Theta_2}\frac{J_m(\eta U)}{J_m(U)}\,,\\
\frac{a_3}{a_1} &= V\frac{\Theta_2\tilde\Theta_2 - m^2\Theta_1\tilde\Theta_1}{\Theta_2\Theta_3}\frac{J_m(\eta U)}{J_m(V)}\,,\\
\frac{a_4}{a_1} &= \left(1 - V\frac{\Theta_2\tilde\Theta_2 - m^2\Theta_1\tilde\Theta_1}{\Theta_2\Theta_3}\right)\frac{J_m(\eta U)}{H_m^{(1)}(V)}\,,\\
\frac{a_5}{a_1} &= 0\,,\\
\frac{a_6}{a_1} &= i\frac{\kappa m\Theta_1}{k\Theta_2}\frac{J_m(\eta U)}{H_m^{(1)}(U)}\,,
\end{aligned}
\tag{A10}
$$

where

$$
\begin{aligned}
\Theta_1 &= \frac{1}{U^2} - \frac{1}{V^2}\,,\\
\tilde\Theta_1 &= \frac{n_\perp^2}{n_{\rm cl}^2}\frac{1}{U^2} - \frac{1}{V^2}\,,\\
\Theta_2 &= \frac{J_m'(U)}{U\,J_m(U)} - \frac{H_m^{(1)\prime}(V)}{V\,H_m^{(1)}(V)}\,,\\
\tilde\Theta_2 &= \frac{n_\parallel^2}{n_{\rm cl}^2}\frac{J_m'(\eta U)}{\eta U\,J_m(\eta U)} - \frac{H_m^{(1)\prime}(V)}{V\,H_m^{(1)}(V)}\,,\\
\Theta_3 &= \frac{J_m'(V)}{J_m(V)} - \frac{H_m^{(1)\prime}(V)}{H_m^{(1)}(V)}\,.
\end{aligned}
\tag{A11}
$$

In the limit $n_\parallel = n_\perp = n_{\rm cl}$, we find $a_3/a_1 = 1$, with all other constants zero. If the core birefringence is small and the fiber is weakly guiding, we expect $|a_3/a_1| \simeq 1$, $|a_5/a_1| = 0$, and $|a_2/a_1|, |a_4/a_1|, |a_6/a_1| \ll 1$, which implies $|E_z| \gg |H_z|$ in the core [see (A4)]; so the ITM modes couple strongly to the bound mode [see the discussion preceding (50)].

5. ITE modes

Combining the boundary conditions with the extra condition $a_3 = 0$ gives a linear set of five equations in five unknowns, with the solution

$$
\begin{aligned}
\frac{a_1}{a_2} &= -i\frac{km\tilde\Theta_1}{\kappa\tilde\Theta_2}\frac{J_m(U)}{J_m(\eta U)}\,,\\
\frac{a_3}{a_2} &= 0\,,\\
\frac{a_4}{a_2} &= -i\frac{km\tilde\Theta_1}{\kappa\tilde\Theta_2}\frac{J_m(U)}{H_m^{(1)}(\eta U)}\,,\\
\frac{a_5}{a_2} &= V\frac{\Theta_2\tilde\Theta_2 - m^2\Theta_1\tilde\Theta_1}{\tilde\Theta_2\Theta_3}\frac{J_m(U)}{J_m(V)}\,,\\
\frac{a_6}{a_2} &= \left(1 - V\frac{\Theta_2\tilde\Theta_2 - m^2\Theta_1\tilde\Theta_1}{\tilde\Theta_2\Theta_3}\right)\frac{J_m(U)}{H_m^{(1)}(V)}\,.
\end{aligned}
\tag{A12}
$$

In the limit $n_\parallel = n_\perp = n_{\rm cl}$, we find $a_5/a_2 = 1$, with all other constants zero. If the core birefringence is small and the fiber is weakly guiding, we expect $|a_5/a_2| \simeq 1$, $|a_3/a_2| = 0$, and $|a_1/a_2|, |a_4/a_2|, |a_6/a_2| \ll 1$, which implies $|H_z| \gg |E_z|$ in the core [see (A4)]; so the ITE modes couple weakly to the bound mode.

6. Normalization

In this paper we assume the core birefringence is small and the fiber is weakly guiding, so we neglect ITE modes. The unbound modes $|\mathbf{E}_s\rangle$ of the text are ITM modes. For the remainder of this Appendix we consider only ITM modes, and, for convenience, take $a_1 = 1$. Then, in the core,

$$
\hat{\mathbf{z}}\cdot\mathbf{E}_s(\mathbf{r}) = J_m(\eta U\rho/R)\,e^{im\phi}\,e^{i\kappa z}\,.
\tag{A13}
$$

This is Eq. (50) of the text.

Our final task is to calculate the normalization integral

$$
\begin{aligned}
\langle\mathbf{E}|\mathbf{D}\rangle &= \int d\mathbf{r}\,\mathbf{E}^*\cdot\mathbf{D}\\
&= 2\pi L\int_0^R d\rho\,\rho\Big(n_\perp^2|E_\rho|^2 + n_\parallel^2|E_\phi|^2 + n_\parallel^2|E_z|^2\Big)\\
&\quad + 2\pi L n_{\rm cl}^2\int_R^{R_\infty} d\rho\,\rho\Big(|E_\rho|^2 + |E_\phi|^2 + |E_z|^2\Big)\,.
\end{aligned}
\tag{A14}
$$

In the limit $R_\infty \to \infty$ the contribution from the core becomes negligible, and we can write

$$
\langle\mathbf{E}|\mathbf{D}\rangle = 2\pi L n_{\rm cl}^2\int_R^{R_\infty} d\rho\,\rho\Big(|E_\rho|^2 + |E_\phi|^2 + |E_z|^2\Big)\,,
\tag{A15}
$$

where the field components are those of the cladding. Consider the E_ρ integral. In the limit $R_\infty \to \infty$

$$
\begin{aligned}
\int_R^{R_\infty} d\rho\,\rho|E_\rho|^2 &= \frac{\kappa^2R^4}{V^4}\int_1^{VR_\infty/R} d\alpha\,\alpha\\
&\quad\times[a_3^*\,J_m'(\alpha) + a_4^*\,H_m^{(1)\prime *}(\alpha)]\\
&\quad\times[a_3\,J_m'(\alpha) + a_4\,H_m^{(1)\prime}(\alpha)]\\
&= \frac{\kappa^2R^3R_\infty}{\pi V^3}\left(|a_3 + a_4|^2 + |a_4|^2\right)\,,
\end{aligned}
\tag{A16}
$$

where we have used the results

$$\int_1^{VR_\infty/R} d\alpha\, \alpha\, J'_m(\alpha)\, J'_m(\alpha) = \frac{VR_\infty}{\pi R}\,,$$

$$\int_1^{VR_\infty/R} d\alpha\, \alpha\, J'_m(\alpha)\, H_m^{(1)'}(\alpha) = \frac{VR_\infty}{\pi R}\,,$$

$$\int_1^{VR_\infty/R} d\alpha\, \alpha\, H_m^{(1)'*}(\alpha)\, H_m^{(1)'}(\alpha) = \frac{2VR_\infty}{\pi R}\,.$$

The E_ϕ and E_z integrals can be evaluated in a similar fashion, giving

$$\int_R^{R_\infty} d\rho\, \rho |E_\phi|^2 = \frac{2k^2 R^3 R_\infty}{\pi V^3} |a_6|^2\,, \tag{A17}$$

$$\int_R^{R_\infty} d\rho\, \rho |E_z|^2 = \frac{RR_\infty}{\pi V}(|a_3 + a_4|^2 + |a_4|^2)\,. \tag{A18}$$

Combining these results and using the relation $\kappa = n_{\rm cl} k \cos\theta_s$ gives

$$\langle \mathbf{E}|\mathbf{D}\rangle = \frac{2LR_\infty n_{\rm cl}}{k\, \sin^3\theta_s}\left(|a_3 + a_4|^2 + |a_4|^2 + \frac{2|a_6|^2}{n_{\rm cl}^2}\right)\,. \tag{A19}$$

If the core birefringence is small and the fiber is weakly guiding, then $a_3 \simeq 1$ and $|a_4|, |a_6| \ll 1$, so

$$\langle \mathbf{E}|\mathbf{D}\rangle \simeq \frac{2LR_\infty n_{\rm cl}}{k\, \sin^3\theta_s}\,. \tag{A20}$$

This is Eq. (48) of the text.

APPENDIX B: CORE-CLADDING INTERFACE FLUCTUATIONS

In this appendix we consider scattering due to thermal fluctuations of the core-cladding interface, and make an order-of-magnitude estimate of the $1/e$ attenuation length μ_i due to these fluctuations. Following Marcuse [24] we represent the locus of the interface by

$$\rho(\phi, z) = R + \epsilon(z)\, \cos(m\phi)\,, \tag{B1}$$

where ρ is the radial coordinate of the interface, ϕ and z are the usual cylindrical coordinates, R is the mean core radius, and ϵ is a small perturbation. The integer m determines the type of interface distortion: $m = 0$ for radius fluctuations, $m = 1$ for bends, $m = 2$ for elliptical profile distortions, and so forth. Here we confine our attention to the $m = 0$ mode, so that

$$\rho(\phi, z) = R + \epsilon(z)\,. \tag{B2}$$

We model the free energy of the core-cladding interface by

$$F_i = \int_0^L dz\, \left\{ \frac{\mathcal{B}}{2}(\rho - R)^2 + \pi\Gamma\rho\left(\frac{\partial\rho}{\partial z}\right)^2 \right\}\,, \tag{B3}$$

where L is the length of the fiber, $\mathcal{B}$ is the bulk modulus of elasticity (assumed the same for the core and cladding), and Γ is the interface surface tension. A calculation analogous to that in Secs. II and III (but much simpler) gives the spatial correlation function

$$\langle \epsilon(z)\epsilon(z')\rangle = \sigma^2\, e^{-|z-z'|/D}\,, \tag{B4}$$

where

$$\sigma^2 = \frac{T}{2\mathcal{B}D} \tag{B5}$$

is the variance, and

$$D = \sqrt{\frac{2\pi R\Gamma}{\mathcal{B}}} \tag{B6}$$

is the correlation length. For correlation functions of this form, and for typical step-index, single-mode, weakly guiding fibers, Marcuse [24] shows that

$$\frac{R^3}{\mu_i \sigma^2} \ll 1\,, \tag{B7}$$

which implies

$$\mu_i \gg \frac{R}{T}\sqrt{8\pi(\mathcal{B}R^3)(\Gamma R^2)}\,. \tag{B8}$$

Marcuse also shows that scattering due to the $m = 1$ and $m = 2$ modes is of the same order of magnitude as for the $m = 0$ mode. High-order modes are suppressed by the surface tension term of the free energy, so (B8) is a reasonable order-of-magnitude estimate of μ_i. With the typical values $R \sim 10^{-6}$ m, $T \sim 4 \times 10^{-21}$ J, $\Gamma \sim 10^{-2}$ J m^{-2} [30,31], and $\mathcal{B} \sim 10^{10}$ J m^{-3}, we find $\mu_i \gg 10^4$ m.

[1] F. J. Baltá-Calleja and A. Peterlin, J. Polym. Sci. A **2**, 1275 (1969).
[2] S. K. Lo, L. M. Galarneau, D. J. Rogers, and S. R. Flom, Mol. Cryst. Liq. Cryst. **201**, 137 (1991).
[3] C. Hu and J. R. Whinnery, J. Opt. Soc. Am. **64**, 1424 (1974).
[4] T. G. Giallorenzi and J. P. Sheridan, J. Appl. Phys. **46**, 1271 (1975).
[5] T. G. Giallorenzi, J. A. Weiss, and J. P. Sheridan, J. Appl. Phys. **47**, 1820 (1976).
[6] J. R. Whinnery, C. Hu, and Y. S. Kwon, IEEE J. Quantum Electron. **13**, 262 (1977).
[7] H. Lin, P. Palffy-Muhoray, and M. A. Lee, Mol. Cryst. Liq. Cryst. **204**, 189 (1991).
[8] J. Fousek and M. Glogarove, Ferroelectrics **53**, 71 (1984).
[9] E. B. Loginov and E. M. Terentjev, Sov. Phys. Crystal-

lography [Kristallographia] **30**, 4 (1985).
[10] N. Schopohl and T. Sluckin, Phys. Rev. Lett. **59**, 2582 (1987).
[11] S. Kralj and T. Sluckin, Phys. Rev. E **48**, R3244 (1993).
[12] C. D. Munzy and N. A. Clark, Phys. Rev. Lett. **68**, 804 (1992).
[13] E. M. Terentjev, J. Phys. (France) II **5**, 159 (1995).
[14] D. W. Cronin, E. M. Terentjev, R. A. Sones, and R. G. Petschek, Mol. Cryst. Liq. Cryst. **238**, 167 (1994).
[15] M. J. Stephen and J. P. Straley, Rev. Mod. Phys. **46**, 686 (1974), where $K_1 = \alpha_{11}$, $K_2 = \alpha_{22}$, $K_3 = \alpha_{33}$, and $K_4 = \alpha_{23}$.
[16] R. B. Meyer, in *Polymer Liquid Crystals*, edited by A. Ciferri, W. Krigbaum, and R. B. Meyer (Academic, New York, 1982).
[17] A. Y. Grosberg and A. V. Zhestkov, Polym. Sci. U.S.S.R. **28**, 97 (1986).
[18] T. Odijk, Liq. Cryst. **1**, 553 (1986).
[19] R. G. Petschek and E. M. Terentjev, Phys. Rev. A **45**, 930 (1992).
[20] R. G. Petschek and E. M. Terentjev (unpublished).
[21] A. Buka and L. Bata, Mol. Cryst. Liq. Cryst. Lett. **49**, 159 (1979).
[22] A. Balogh, N. Éber, and L. Bata, Mol. Cryst. Liq. Cryst. **200**, 29 (1991).
[23] P. G. de Gennes and J. Proust, *The Physics of Liquid Crystals*, 2nd ed. (Clarendon Press, Oxford, 1993).
[24] D. Marcuse, *Theory of Dielectric Optical Waveguides* (Academic, Boston, 1991), Chap. 2.
[25] W. B. Allan, *Fibre Optics: Theory and Practice* (Plenum, London, 1973), Chap. 9.
[26] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[27] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950).
[28] A. W. Snyder, IEEE Trans. Microwave Theory Tech. **19**, 720 (1971).
[29] A. W. Snyder and J. D. Love, *Optical Waveguide Theory* (Chapman and Hall, London, 1983).
[30] C. I. Poser and I. C. Sanchez, Macromolecules **14**, 361 (1981).
[31] A. K. George and K. P. Mohandas, J. Phys. Condens. Matter **4**, 7691 (1992).